

A model for the emergence of thermal plumes in Rayleigh–Bénard convection at infinite Prandtl number

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We propose a two-dimensional model of three-dimensional Rayleigh–Bénard convection in the limit of very high Prandtl number and Rayleigh number, as in the Earth's mantle. The model equation describes the evolution of the first moment of the temperature anomaly in the thermal boundary layer, which is assumed thin with respect to the scale of motion. This two-dimensional field is transported by the velocity that it induces and is amplified by surface divergence. This model explains the emergence of thermal plumes, which arise as finite-time singularities. We determine critical exponents for these singularities. Using a smoothing method we go beyond the singularity and reach a stage of developed convection. We describe a process of plume merging, leaving room for the birth of new instabilities. The heat flow at the surface predicted by our two-dimensional model is found to be in good agreement with available data.

1. Introduction

Thermal plumes are ubiquitous in convection at high Rayleigh number, far from the threshold of instability (e.g. Nataf 1991; Siggia 1994). The development of a convective plume from a localized source is a well-known process, and self-similar solutions of the equations of motion are available (Batchelor 1954; Sparrow, Husar & Goldstein 1970; Moses, Zocchi & Libchaber 1993; Olson, Schubert & Anderson 1993). However the *emergence* of a plume from a uniformly heated surface is not clearly understood. Plumes are strongly nonlinear structures, out of reach of perturbative methods on the instability modes. Modelling the statistics of plume production and interactions is a challenge, and it is a central issue for predicting the heat flux average and its variability.

The aim of the present paper is to propose a model for the emergence of convective plumes and their further interactions. Our main assumptions are that the fluid interior is well mixed, and temperature anomalies are restricted to a thermal boundary layer, which is thin in comparison with horizontal scales. This condition can be always satisfied for an appropriate initial condition: starting from a fluid at uniform temperature, we suddenly impose a different surface temperature θ_s (with a perturbation at large scale to initiate the instability). Then temperature diffuses within a thin boundary layer until convection is initiated, and thermal plumes emerge as singularities arising after a finite time in our boundary layer approximation.

Then the approximation breaks down: plumes are fully developed and feed the interior with temperature anomalies. However, we shall still capture the main features

of the dynamics with an appropriate phenomenological smoothing of the singularities. The further plume interactions will be described and the mean heat flux will be obtained by a numerical model with reduced dimensionality: a two-dimensional model on the horizontal heated surface represents the three-dimensional convection.

The development of a thermal plume is driven by the nonlinearity in the heat transport equation, rather than by hydrodynamic turbulence, so this phenomenon can be conveniently analysed in the limit of high Prandtl number, for which inertia is neglected. We therefore restrict our study to this limit of high Prandtl number, and we furthermore assume a free surface condition.

This convection regime ($Ra \gg 1$, $Re \ll 1$ and $Pr \gg 1$) corresponds to that of the silicated parts of Earth-like planetary interiors (e.g. Bercovici, Ricard & Richards 1998); as an example, for the Earth's mantle, i.e. the outer 3000 km of our planet, $Ra = 10^8$, $Re = 10^{-15}$ and $Pr = 10^{23}$.

In such planetary interiors, the viscosity increases by many orders of magnitude in the relatively thin upper layers, due to the lower temperature. Describing this rapid variation of viscosity with depth is a severe difficulty for full three-dimensional convective models. In our boundary layer model, this effect shows up instead as a viscosity of the 'surface skin'. Various rheologies of this surface skin could be implemented as well, accounting for the formation of rigid plates with complex fracturation properties, which are still out of reach of current convection models.

The general idea of this paper is to reduce the dimensionality of a convective system by integration of the dynamic equations across the boundary layers. Such a procedure has been fruitfully used to investigate the dynamics of bubbles (e.g. Pozrikidis 1992) or the situations more akin to thermal convection such as Rayleigh–Taylor instability (Canright & Morris 1993) and Marangoni convection (Thess, Spirn & Jüttner 1997). It is also a classical method in the geophysical literature to study the equilibrium of tectonic plates (Vilotte & Daignières 1982; England & McKenzie 1982; Houseman & England 1986; Bird 1988). However, the application to Rayleigh–Bénard convection where both the mechanical and thermal equations are integrated has never been done before and allows us to describe mathematically the destabilization of the thermal boundary layers qualitatively explained by Howard (1966).

In next section, we present our boundary layer model. We first derive general relationships valid for any rheology, and then specify the model for a Newtonian fluid. We find that the effect of buoyancy in the thermal boundary layer results in a horizontal stress acting on the interior flow. This stress has formal similarities with surface tension effects in Marangoni convection, as analysed by Thess *et al.* (1997). In their model, the surface temperature field induces a surface velocity field, obtained by solving the Stokes problem in the interior, with the boundary condition given by the viscous stress at the surface. Therefore the surface temperature is transported by a velocity which depends linearly, but non-locally, on the temperature field. The 'closure relationship' relating the surface velocity to temperature has a simple expression in Fourier space (but is non-local in real space). We find the same closure relationship as in the Marangoni case, but the active quantity is the first moment of the temperature in the boundary layer, instead of the surface temperature. This quantity is transported like temperature, but with an additional production term, which leads to the onset of singularities with infinite values, representing the thermal plumes. In Marangoni convection, the temperature instead remains finite, but its spatial derivatives become singular.

In §3, we study the initial growth of our boundary layer by a linear stability analysis. In §4 we discuss in detail the closure relationship giving the velocity field induced by different temperature fields. The properties of the singularities generated

in the model is discussed in §5. Finally in §6, we propose a simple phenomenological smoothing of the singularities, allowing a permanent convective regime to be reached. Plumes (smoothed singularities) are observed to merge under the effect of their induced velocity field, and new plumes are formed between existing ones. We extend the model to two dimensions, representing three-dimensional convection, with the formation of convective lines. The averaged heat flux is also calculated, and found to be in good agreement with numerical results in the literature.

2. A two-dimensional model of three-dimensional Rayleigh–Bénard convection

2.1. The three-dimensional model

We study thermal convection driven by buoyancy forces in the Boussinesq approximation. We assume that the Prandtl number, i.e. the ratio between the kinematic viscosity ν and thermal diffusivity κ , is very large so that the Reynolds number is very small. In these conditions, the equations controlling the dynamics are

$$\nabla \cdot \mathbf{v} = 0, \quad (2.1)$$

$$\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g} = 0, \quad (2.2)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{v} \cdot \nabla) \theta = \nabla \cdot (\kappa \nabla \theta). \quad (2.3)$$

They express respectively the conservation of mass, the balance of forces (neglecting inertia), and the heat transport. $\boldsymbol{\tau}$ denotes the total stress tensor, which will be related to the velocity \mathbf{v} by an appropriate rheology, and \mathbf{g} the acceleration due to gravity. Equations (2.2) and (2.3) are coupled as the density ρ varies with temperature θ as $\rho = \rho_0(1 - \alpha\theta)$, with $\alpha\theta \ll 1$.

We consider for simplicity a uniform gravity (although our method readily extends to a spherical geometry). The surface is supposed infinite in the horizontal directions x and y . The z -axis is directed downward and the convective system extends infinitely in the direction of positive z . The motion is driven by temperature anomalies near the surface, while the deep interior is supposed at uniform temperature. We choose this temperature as the reference temperature, so that $\theta \rightarrow 0$ for $z \rightarrow +\infty$ and θ reaches θ_S , which is negative, at the surface. Furthermore, since the motion is damped in the deep interior ($z \rightarrow +\infty$), the stress tensor reduces asymptotically to a pressure p_0 , $\tau_{ij} \rightarrow -p_0\delta_{ij}$, and equation (2.2) reduces to the hydrostatic relation $\nabla p_0 = \rho_0 \mathbf{g}$, or $p_0 = \rho_0 g z$. The origin of the coordinate z is chosen in order to cancel the constant of integration which should appear in this expression. In other words, the free surface is at $z = 0$ in the reference state with $\theta = 0$ everywhere. We express the stress tensor $\boldsymbol{\tau}$ as the sum of the reference hydrostatic pressure and a stress $\boldsymbol{\tau}'$, which is driven by the temperature heterogeneity θ and vanishes at large depth,

$$\tau_{ij} = \tau'_{ij} - \rho_0 g z \delta_{ij}. \quad (2.4)$$

In summary, the conditions in the deep interior are

$$\mathbf{v} \rightarrow 0, \quad \tau'_{ij} \rightarrow 0, \quad \theta \rightarrow 0 \quad \text{for } z \rightarrow +\infty. \quad (2.5)$$

For motion with sufficiently large horizontal scales, the stress $\boldsymbol{\tau}'$ reduces to the hydrostatic pressure associated with the temperature heterogeneity θ . Its typical value is $\rho_0 g \alpha \theta H$, where H is the thickness of the thermal boundary layer, much smaller

than the horizontal scale of motion L . The hydrostatic balance provides in all cases a first-order estimate of the stress $\boldsymbol{\tau}'$,

$$|\boldsymbol{\tau}'| = o(\rho_0 g \alpha \theta H) \ll \rho_0 g H. \quad (2.6)$$

Owing to the temperature heterogeneity θ , the free surface is slightly deformed, to a ‘depth’ $z = -h(x, y)$, much smaller than H , and the normal \mathbf{n} to the surface is nearly vertical. The free surface condition imposes that the normal components of the total stress tensor vanish, $\boldsymbol{\tau}' \cdot \mathbf{n} + \rho_0 g h \mathbf{n} = 0$. With the estimate (2.6) for $|\boldsymbol{\tau}'|$, it results that $h/H \sim \alpha \theta$. We can therefore take the stress tensor $\boldsymbol{\tau}'$ at $z = 0$ for the free surface condition, within an error of order $(\alpha \theta)^2$ (as estimated by linearizing $\boldsymbol{\tau}'$ with respect to z). Furthermore, the slope n_x/n_z (or n_y/n_z) is of order $h/L = \alpha \theta H/L$, and it can be safely neglected, so we can write the free surface conditions as

$$\tau'_{xz}(x, y, 0) = \tau'_{yz}(x, y, 0) = 0, \quad (2.7)$$

$$\tau'_{zz}(x, y, 0) = -\rho_0 g h(x, y). \quad (2.8)$$

This provides a free slip condition at $z = 0$ for the tangential stresses, while the normal component τ'_{zz} determines the weak topography $h(x, y)$. In addition to these dynamical conditions, we have the kinematic condition for a material surface, which reduces to

$$v_z(x, y, 0) = 0 \quad (2.9)$$

within our approximations.

In summary, the three components of the equation of motion (2.2) are

$$\partial_x \tau'_{xx} + \partial_y \tau'_{xy} + \partial_z \tau'_{xz} = 0, \quad (2.10)$$

$$\partial_x \tau'_{yx} + \partial_y \tau'_{yy} + \partial_z \tau'_{yz} = 0, \quad (2.11)$$

$$\partial_x \tau'_{zx} + \partial_y \tau'_{zy} + \partial_z \tau'_{zz} = \rho_0 \alpha g \theta. \quad (2.12)$$

They must be solved together with (2.1) and (2.3), using the appropriate rheology, and with the boundary conditions (2.5), (2.7), (2.8) and (2.9).

2.2. Integration of the stress across the boundary layer

The thermal boundary layer entrains the interior like a ‘skin’ driven by gravity effects, and this process can be described by integrating the equations of motion across the boundary layer. We denote \bar{X} the z -integrated value of a quantity X , from $z = 0$ to the depth Z . We first get exact equations, but will then assume that Z is much smaller than the horizontal scale of motion L , while beyond the boundary layer (where $\theta \approx 0$), $H \lesssim Z \ll L$.

Such a vertical integration, applied to the vertical component of the equation of motion (2.12), yields, using the free surface condition (2.8),

$$\partial_x \bar{\tau}'_{zx} + \partial_y \bar{\tau}'_{zy} + \tau'_{zz} + \rho_0 g h = \rho_0 g \alpha \bar{\theta}, \quad (2.13)$$

giving the topography $-h(x, y)$ from the dynamical variables. Applying this relation to the deep interior, $z \rightarrow +\infty$, τ'_{zz} vanishes. For sufficiently large scales, the horizontal derivatives become negligible, and (2.13) reduces to $h = \alpha \bar{\theta}$, which simply expresses the Archimedes principle, also called isostasy among geophysicists.

To get a constraint on the dynamical variables, we multiply all terms of (2.12) by z before the vertical integration, yielding

$$\partial_x \overline{z \tau'_{zx}} + \partial_y \overline{z \tau'_{zy}} + Z \tau'_{zz} - \bar{\tau}'_{zz} = \rho_0 g \alpha \overline{z \theta} \quad (2.14)$$

(where we have used an integration by parts for the term in τ'_{zz}). The last term of the left-hand side dominates the three first terms. Therefore, relation (2.14) relates the average vertical stress in the boundary layer to the first moment of the temperature. This behaviour has already been emphasized in the geophysical literature and equation (2.14) has been sometimes called ‘stress moment law’ (Fleitout & Froidevaux 1982, 1983; Ricard, Fleitout & Froidevaux 1984).

We similarly integrate the horizontal components (2.10) and (2.11) of the momentum equation, using the free surface condition (2.7),

$$\partial_x \overline{\tau'_{xx}} - \overline{\tau'_{zz}} + \partial_y \overline{\tau'_{xy}} + \tau'_{xz} = -\partial_x \overline{\tau'_{zz}}, \quad (2.15)$$

$$\partial_x \overline{\tau'_{xy}} + \partial_y \overline{\tau'_{yy}} - \overline{\tau'_{zz}} + \tau'_{yz} = -\partial_y \overline{\tau'_{zz}}. \quad (2.16)$$

We have written the two equations so that their left-hand sides depend only on the deviatoric part of the stress tensor, which will be related to the velocity once the rheology is specified (in other words the pressure term has been eliminated on the left-hand side).

The right-hand sides of (2.15) and (2.16) can be related by (2.14) to the temperature moment, which we define as

$$M \equiv -\overline{z\theta} \quad (2.17)$$

$$\equiv -\int_0^\infty z\theta \, dz, \quad (2.18)$$

the minus sign is introduced to make M positive in the convection problem. Then (2.15) becomes

$$\tau'_{xz} + Z\partial_x \tau'_{zz} + \partial_x \overline{\tau'_{xx} - \tau'_{zz}} + \partial_y \overline{\tau'_{xy}} + \partial_x \partial_x \overline{z\tau'_{zx}} + \partial_x \partial_y \overline{z\tau'_{zy}} = -\rho_0 g \alpha \partial_x M. \quad (2.19)$$

All terms on the left-hand side depend only on the deviatoric part of the stress tensor, except the second term $Z\partial_x \tau'_{zz}$. We can rearrange this term by writing $\partial_x \tau'_{zz} = \partial_x (\tau'_{zz} - \tau'_{xx}) + \partial_x \tau'_{xx}$, and, using (2.10), $\partial_x \tau'_{xx} = -\partial_y \tau'_{xy} - \partial_z \tau'_{xz}$. Introducing this in (2.19), and repeating the same procedure for the y -component, (2.15) and (2.16) transform into

$$\begin{aligned} \tau'_{xz} - Z\partial_z \tau'_{xz} + \partial_x [\overline{\tau'_{xx} - \tau'_{zz}} - Z(\tau'_{xx} - \tau'_{zz})] + \partial_y [\overline{\tau'_{xy}} - Z\tau'_{xy}] \\ + \partial_x \partial_x \overline{z\tau'_{zx}} + \partial_x \partial_y \overline{z\tau'_{zy}} = -\rho_0 g \alpha \partial_x M, \end{aligned} \quad (2.20)$$

$$\begin{aligned} \tau'_{yz} - Z\partial_z \tau'_{yz} + \partial_y [\overline{\tau'_{yy} - \tau'_{zz}} - Z(\tau'_{yy} - \tau'_{zz})] + \partial_x [\overline{\tau'_{xy}} - Z\tau'_{xy}] \\ + \partial_y \partial_y \overline{z\tau'_{zy}} + \partial_x \partial_y \overline{z\tau'_{zx}} = -\rho_0 g \alpha \partial_y M, \end{aligned} \quad (2.21)$$

which provide exact relations between the deviatoric part of the stress tensor, integrated over the ordinate range $[0, Z]$, and the first temperature moment M .

The left-hand side of these equations is clearly dominated by the first term, the other terms bringing corrections with relative magnitude Z/L and $(Z/L)^2$ (remembering that $\overline{X} \sim ZX$). Thus (2.20) and (2.21) state that at first order the thermal boundary layer drives the interior flow with a horizontal surface stress (τ'_{xz}, τ'_{yz}) proportional to the gradient of M on the surface. This is analogous to the free surface condition in Marangoni convection, for which M should be replaced by the surface temperature. Furthermore, we shall see in §2.4 that M is advected by the horizontal flow, like a temperature (however there is an additional production term for M , proportional

to the horizontal flow divergence, so the analogy with Marangoni convection is not exact).

Before proceeding further, it is useful to consider corrections of order Z/L on the left-hand side of (2.20) and (2.21). The second term $-Z\partial_z\tau'_{xz}$ can be viewed as a correction to linearly extrapolate τ'_{xz} from its value at depth $z = Z$ to $z = 0$. Therefore the boundary layer really acts as a surface skin at position $z = 0$, instead of the arbitrary position Z . The next two terms are related to horizontal shear effects, describing an horizontal viscosity of the surface skin. Finally, the last two terms are clearly of order $(Z/L)^2$ with respect to τ'_{xz} or τ'_{yz} , and can be neglected.

To get explicit results, we now assume a Newtonian rheology, with a viscosity η ,

$$\tau'_{ij} = \eta(\partial_i v_j + \partial_j v_i) - p\delta_{ij}. \quad (2.22)$$

This viscosity is possibly non-uniform in the boundary layer, beyond which it reaches a uniform value η_0 . We introduce the relative excess ‘surface’ viscosity

$$\sigma = \int_0^Z \frac{\eta(z) - \eta_0}{\eta_0} dz, \quad (2.23)$$

which becomes independent of the upper bound Z when it is beyond the boundary layer (since the integrand tends to 0). Introduction of this rheology in (2.20) and (2.21), expressing $\partial_z v_z = -\partial_x v_x - \partial_y v_y$ from the incompressibility condition, and considering that v_x and v_y cannot depend on z over the boundary layer at first order, we get

$$-\partial_z v_x - 2\partial_x[\sigma(2\partial_x v_x + \partial_y v_y)] - \partial_y[\sigma(\partial_x v_y + \partial_y v_x)] = \frac{\rho_0 g \alpha}{\eta_0} \partial_x M, \quad (2.24)$$

$$-\partial_z v_y - 2\partial_y[\sigma(2\partial_y v_y + \partial_x v_x)] - \partial_x[\sigma(\partial_x v_y + \partial_y v_x)] = \frac{\rho_0 g \alpha}{\eta_0} \partial_y M. \quad (2.25)$$

The first terms of these two equations, $\partial_z v_x$ and $\partial_z v_y$, provide the stresses transmitted to the interior, i.e. the z -derivatives of the horizontal velocities for the internal flow taken at $z = 0$. These stresses are non-zero even though v_x and v_y in the boundary layer do not depend on z at first order.

As is usual in fluid dynamics, we can decompose the horizontal strain into a traceless strain tensor \mathbf{T} and a horizontal divergence,

$$\mathbf{T}_{xx} = -\mathbf{T}_{yy} = \partial_x v_x - \partial_y v_y, \quad (2.26)$$

$$\mathbf{T}_{xy} = \mathbf{T}_{yx} = \partial_x v_y + \partial_y v_x. \quad (2.27)$$

Then, denoting the horizontal velocity vector by \mathbf{v}_H and the horizontal nabla operator by ∇_H , equations (2.24) and (2.25) can be written as

$$-\partial_z \mathbf{v}_H - \nabla_H \cdot (\sigma \mathbf{T}) - \nabla_H (3\sigma \nabla_H \cdot \mathbf{v}_H) = \frac{\rho_0 g \alpha}{\eta_0} \nabla_H M. \quad (2.28)$$

It provides boundary conditions at $z \rightarrow 0$ for the interior flow. The effect of the boundary layer on the interior is therefore equivalent to a surface skin providing a stress proportional to the horizontal gradient of M , with a two-dimensional shear viscosity $\sigma\eta_0$ and a compressional viscosity $3\sigma\eta_0$.

2.3. The interior flow

In the interior, the temperature heterogeneity θ vanishes, and we assume a constant viscosity η_0 , so that the momentum equation (2.2) reduces to the Stokes equation,

$$\eta_0 \nabla^2 \mathbf{v} - \nabla p = 0. \quad (2.29)$$

This equation must be solved, together with the incompressibility equation (2.1), with the conditions of decaying motion at $z \rightarrow +\infty$, and the boundary conditions at $z = 0$ provided by (2.28).

This problem is classically solved (see Chandrasekhar 1981) in terms of the vertical velocity v_z and vertical vorticity $\Omega_z = \partial_x v_y - \partial_y v_x$. These two quantities determine the poloidal and toroidal parts of the velocity field respectively, i.e. the Helmholtz decomposition of the horizontal velocity projection \mathbf{v}_H in each horizontal plane,

$$\mathbf{v}_H = \nabla_H \phi - \mathbf{e}_z \times \nabla_H \psi, \quad (2.30)$$

where the first term is irrotational and the second term is non-divergent (\mathbf{e}_z is the vertical unit vector). The two scalars ϕ and ψ are obtained from v_z and Ω_z by solving the Poisson equations, obtained by taking the horizontal divergence and the curl of (2.30) respectively,

$$\nabla_H^2 \phi = -\partial_z v_z, \quad (2.31)$$

$$\nabla_H^2 \psi = -\Omega_z. \quad (2.32)$$

For a velocity field with a harmonic horizontal dependence, $\mathbf{v} = \hat{\mathbf{v}}(z)e^{i\mathbf{k}\cdot\mathbf{r}}$ with $\mathbf{k} = (k_x, k_y)$, these relations yield (with $k \equiv |\mathbf{k}|$),

$$\hat{v}_x = \frac{i}{k^2} \left(k_x \frac{d\hat{v}_z}{dz} + k_y \hat{\Omega}_z \right), \quad (2.33)$$

$$\hat{v}_y = \frac{i}{k^2} \left(k_y \frac{d\hat{v}_z}{dz} - k_x \hat{\Omega}_z \right). \quad (2.34)$$

Taking the curl of (2.29) eliminates the pressure and yields $\nabla^2 \Omega_z = 0$, or equivalently

$$(\partial_z^2 + \nabla_H^2) \Omega_z = 0. \quad (2.35)$$

Taking the divergence of (2.29) yields $\nabla^2 p = 0$ (taking into account the flow incompressibility). Then taking the Laplacian of (2.29) yields $\nabla^4 \mathbf{v} = 0$, whose z -component is

$$(\partial_z^2 + \nabla_H^2)^2 v_z = 0. \quad (2.36)$$

Solutions with a sine wave horizontal dependence of (2.35) and (2.36), which vanish for $z \rightarrow +\infty$, are respectively,

$$\hat{v}_z = uz \exp(-kz), \quad (2.37)$$

$$\hat{\Omega}_z = w \exp(-kz). \quad (2.38)$$

Then the horizontal velocity components \hat{v}_x and \hat{v}_y are obtained from (2.33) and (2.34).

The boundary condition (2.28) at $z = 0$ is easily taken into account in the case of a uniform surface viscosity σ . Taking the horizontal divergence and curl of (2.28) then yield respectively

$$\eta_0(\partial_z + 4\sigma\nabla_H^2)\partial_z v_z = \rho_0 g \alpha \nabla_H^2 M, \quad (2.39)$$

$$(\partial_z + \sigma\nabla_H^2)\Omega_z = 0. \quad (2.40)$$

Therefore the toroidal mode is not excited by convection for σ uniform ($\Omega_z = 0$ everywhere). Introducing expression (2.37) into (2.39) yields $2\eta_0(1+2\sigma k)u = \text{sign}(k)\rho_0 g \alpha k M$. The resulting relationship between the surface velocity and the Fourier transform $\hat{M}(\mathbf{k})$

of the temperature moment can be written

$$\hat{\mathbf{v}}_H(k) = i \frac{\mathbf{k}}{2k} \frac{1}{1 + 2\sigma k} \frac{\rho_0 g \alpha}{\eta_0} \hat{M}(\mathbf{k}). \quad (2.41)$$

In this section we have assumed that below the thermal boundary layer the viscosity is uniform. However it would have been straightforward to develop a model with a vertically stratified viscosity at depth. For instance, if the interior is layered, equations (2.35) and (2.36) remain valid in each layer, but continuity of velocities and stresses must be imposed at each interface. Whatever the viscosity stratification, we would have obtained a linear relationship between the Fourier components of the surface velocity and those of the temperature moment with no excitation of toroidal motions. Thus, with minor modifications, our approach could be applied to planetary interiors where the viscosity is likely to increase with depth.

2.4. The transport of the first temperature moment

Since temperature acts only through its first moment M , it is useful to get an evolution equation for this quantity by taking the first moment of (2.3),

$$\frac{\partial M}{\partial t} + (\mathbf{v}_H \cdot \nabla_H) M - \int_0^{+\infty} z v_z \frac{\partial \theta}{\partial z} dz = \nabla_H (\kappa \nabla_H M) - \int_0^{+\infty} z \frac{\partial}{\partial z} \left(\kappa \frac{\partial \theta}{\partial z} \right) dz. \quad (2.42)$$

In this equation we have permuted \mathbf{v}_H and the vertical integration, assuming that \mathbf{v}_H does not depend on z in the boundary layer, which is valid at order (H/L) . This assumption also implies, by integration of the mass conservation equation (2.1), that $v_z = -z \nabla_H \cdot \mathbf{v}_H$. Thus, using integrations by parts, and recalling that both θ and $\partial \theta / \partial z$ tend to zero for $z \rightarrow +\infty$, and that $\theta = \theta_s$ at $z = 0$, we transform (2.42) into

$$\frac{\partial M}{\partial t} + (\mathbf{v}_H \cdot \nabla_H) M + 2M \nabla_H \cdot \mathbf{v}_H = \kappa \nabla_H^2 M - \kappa \theta_s \quad (2.43)$$

(assuming κ constant). Therefore we have transformed a three-dimensional problem with four unknowns, θ and the three components of \mathbf{v} (equations (2.1), (2.2), (2.3)) into a two-dimensional problem with three unknowns, M and the two components of \mathbf{v}_H . To close the system we need to relate \mathbf{v}_H to M , using (2.41), which will now be referred to as the ‘closure law’.

To elucidate the physical meaning of M , we assume that the temperature is simply described by $\theta = \theta_s \operatorname{erfc}(z/H(x, y, t))$, where erfc is the complementary error function, and H the thickness of the thermal boundary layer. Introduction of this temperature dependence in the definition of M (2.18) leads to

$$M(x, y, t) = -\frac{1}{4} \theta_s H^2(x, y, t), \quad (2.44)$$

which shows that $(M)^{1/2}$ is a measure of the thermal boundary thickness.

The first term of equation (2.43) is equal to $2(M)^{1/2}(\partial(M)^{1/2}/\partial t)$ and the second and third terms can be combined together as $2(M)^{1/2} \nabla_H((M)^{1/2} \mathbf{v}_H)$ so that equation (2.43) can be expressed as a transport equation for the quantity $(M)^{1/2}$.

By integration of this transport equation over the whole surface, we get

$$\frac{d}{dt} \iint (M)^{1/2} d^2 \mathbf{r} = \frac{\kappa}{4} \iint \frac{(\nabla_H M)^2}{(M)^{3/2}} d^2 \mathbf{r} - \frac{\kappa}{2} \iint \frac{\theta_s}{(M)^{1/2}} d^2 \mathbf{r}. \quad (2.45)$$

We remark that the left-hand side is conserved when κ is zero. This is not surprising as $(M)^{1/2}$ is proportional to the thickness of the boundary layer (2.44) and thus (2.45) expresses the conservation of the boundary layer volume in the absence of diffusion.

The thermal diffusion increases the volume of the boundary layer by cooling at the surface, especially when the boundary layer is thin (the term $-\theta_S/(M)^{1/2}$ is large); and by lateral diffusion acting on the boundary layer undulations.

In the case $\sigma = 0$, the closure law reduces to $\hat{\mathbf{v}}_H = i(\mathbf{k}/2k)\hat{M}$. This is analogous to the result of Thess *et al.* (1997) for Marangoni convection, where M would be replaced by the surface temperature. However equation (2.43) differs from a usual transport equation by the term $2M\nabla_H \cdot \mathbf{v}_H$. When instability develops, this term will be a strong source of M , leading to a peak with diverging M , corresponding to the emergence of a thermal plume. Thess *et al.* (1997) also find the development of singularities in Marangoni convection, but the transported quantity (temperature) remains bounded.

3. Stability analysis

3.1. Non-dimensionalization

It is convenient to get a non-dimensional version of our dynamical model, defining a length scale D by

$$D^3 = -\frac{\eta_0 \kappa}{\rho_0 g \alpha \theta_S} \quad (3.1)$$

(θ_S is negative). This is the thickness for which the Rayleigh number, based on the vertical temperature difference θ_S , is unity. The time is then scaled by the diffusive time scale (D^2/κ) and the temperature moment M by $-\theta_S D^2$. Using parameters applicable for the Earth D would be of order 10 km, the time scale about 3 Myrs and the moment scale about 1.5×10^5 K km². With this change of variables we obtain

$$\frac{\partial M}{\partial t} + (\mathbf{v}_H \cdot \nabla_H)M + 2M\nabla_H \cdot \mathbf{v}_H = \nabla^2 M + 1, \quad (3.2)$$

with

$$\hat{\mathbf{v}}_H(\mathbf{k}) = i \frac{\mathbf{k}}{2k} \frac{\hat{M}(\mathbf{k})}{1 + 2\sigma k}, \quad (3.3)$$

where for simplicity of notation we have kept the same symbols for the new quantities which are now without units. In the case $\sigma = 0$, the equation becomes universal, i.e. without any parameters.

We have written a numerical code to solve (3.2) and (3.3). This pseudo-spectral code uses a fast Fourier transform with at most $(512)^2$ points in one or two dimensions and assumes periodicity. This code has been parallelized on 16 processors. As an initial condition we assume that M , either one- or two-dimensional, consists of a small-amplitude white noise with a positive small average value (M is everywhere positive). The moment M always follows the typical evolution depicted in figure 1. The average value of M , $\langle M \rangle$ increases linearly with time and its perturbation $\max |M - \langle M \rangle|$ starts to decrease (diffusive regime) then increases slowly (marginal growth) and reaches a finite-time singularity. The duration of each period depends on the initial conditions and the values of σ .

3.2. Marginal stability

A first insight into the behaviour of our boundary layer model of Rayleigh–Bénard convection will be given by a marginal stability analysis. Let us assume that $M(x, t) = M_0(t) + m(k, t) \cos(kx)$ with $|m| \ll |M_0|$ and $M_0(0) = 0$, so that we start with a viscous half-space at zero temperature on which we suddenly impose a negative surface

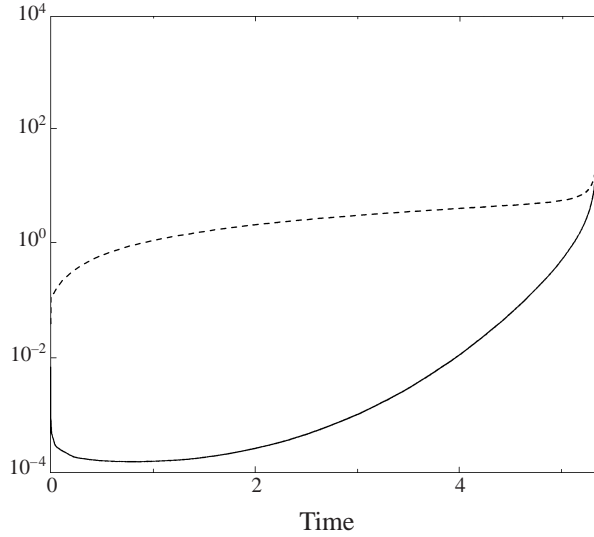


FIGURE 1. Typical evolution of the averaged moment $\langle M \rangle$ (dashed line) and the maximum of its fluctuation $\max |M - \langle M \rangle|$ (solid line). $\langle M \rangle$ increases linearly with time at the beginning (which is not obvious in this linear-logarithmic plot). Three phases are seen for the fluctuations: a decrease, a slow increase, and a finite-time instability. In this simulation we have assumed $\sigma = 0$, but qualitatively the same behaviour is observed for non-zero σ .

temperature. For simplicity, we first assume that $\sigma = 0$. By plugging the expression for M into equations (3.2) and (3.3), linearized with respect to the small amplitude m , we get

$$M_0(t) = t, \quad (3.4)$$

$$\frac{\partial m}{\partial t} = k(t - k)m. \quad (3.5)$$

The growth-rate factor of (3.5) is depicted in figure 2 at a given time. Equation (3.5) shows that all wavenumbers between $k = 0$ and $k = t$ are unstable. The most unstable is the wavenumber $k_m = t/2$. The destabilization of the system thus starts at long wavelength. This justifies our long-wavelength approximation, at least in the initial stage of the evolution.

Recording that (in real units) $M_0 \sim -\frac{1}{4}\theta_S H^2$ (2.44) where H is the thickness of the thermal boundary layer, equation (3.4) simply states in real units that

$$H^2 = 4\kappa t, \quad (3.6)$$

which just expresses the diffusive growth of the thermal boundary layer. The selection of the most unstable wavenumber can be rewritten

$$k_m H = -\frac{1}{4} \frac{\rho_0 \alpha g \theta_S H^3}{\eta_0 \kappa}, \quad (3.7)$$

stating that the most unstable wavenumber, normalized by the boundary layer thickness, is one fourth of the local Rayleigh number (which increases with $t^{3/2}$).

Equation (3.5) can be easily integrated, for a given k :

$$m(t) = m_0 \exp [kt(\frac{1}{2}t - k)]. \quad (3.8)$$

The amplitude at any wavenumber first decreases by thermal diffusion then increases

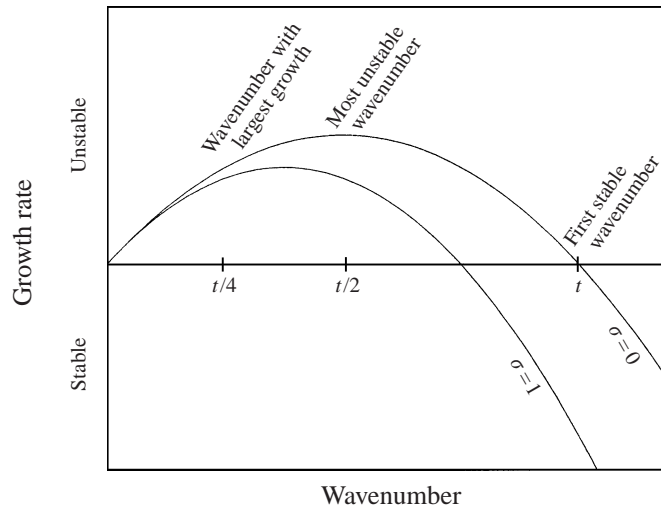


FIGURE 2. Growth rate as a function of wavenumber (at a given time t) for a uniform-viscosity fluid ($\sigma = 0$) or with a viscous lid ($\sigma = 1$).

when the boundary layer is sufficiently thick to sustain the instability. At a given time, the wavelength perturbation that has grown most strongly corresponds to $k = t/4$. Assuming that in the white-noise initial conditions all wavenumbers had equal amplitudes, the perturbation maximum is roughly controlled by the wavenumber that has grown the most; therefore,

$$\max m(t) \sim m_0 \exp\left(\frac{t^3}{16}\right), \tag{3.9}$$

or in real units

$$\max m(t) \sim m_0 \exp\left(\frac{1}{128} \frac{H^6}{D^6}\right). \tag{3.10}$$

This linear analysis is valid as long as $|m(t)| \ll M_0(t)$, i.e. as long as in figure 1 the continuous line ($|m(t)|$) stays below the dashed line ($M_0(t)$).

We can verify numerically these analytic solutions. Figure 3 represents the same data set as figure 1, $\max_x[M(x,t)]$, but this time as a function of t^3 . The numerical solution shows an excellent fit to the analytical expression.

In the case $\sigma \neq 0$, when additional viscous effects are present in the boundary layer, this marginal stability analysis has to be somewhat modified. The instantaneous growth rate is decreased together with the range of unstable wavevectors (see figure 2). Equation (3.9) has to be corrected and one gets at first order

$$m(t) = m_0 \exp\left(\frac{t^3}{16} \frac{1}{1 + \sigma t}\right). \tag{3.11}$$

This instability growth is therefore slowed down by the effect of a viscous lid and we see numerically that the time for the singularity to occur increases. Although equation (3.11) is qualitatively in agreement with the numerical experiments, quantitatively the agreement is poor as σt becomes rapidly of order unity and a higher-order expansion should be done.

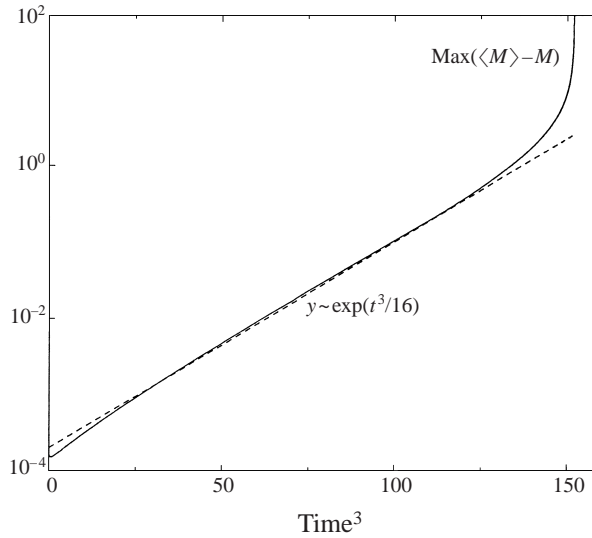


FIGURE 3. $\text{Max}|M - \langle M \rangle|$ as a function of t^3 . Starting from a white noise, the slope is $1/16$ according to equation (3.9).

4. The closure relationship

4.1. Back in real space

From figure 1 or 3, it seems obvious that the system exhibits finite-time singularities. In order to describe what happens closer to this singularity, we must study in the real space our closure relationship (3.3). The multiplication of Fourier components corresponds to a convolution product in real space assuming that the surface of the fluid is unbounded

$$\mathbf{v}_H(\mathbf{r}) = \iint \mathbf{K}(\mathbf{r} - \mathbf{r}')M(\mathbf{r}') d^2\mathbf{r}', \tag{4.1}$$

where $\mathbf{K}(\mathbf{r})$ is the Fourier transform of $i(\mathbf{k}/2k)(1 + 2\sigma k)^{-1}$. We will consider the two limiting cases, $\sigma = 0$, and $\sigma \gg 1$.

In the first case, one has $\hat{\mathbf{v}}_H = i(\mathbf{k}/2k)\hat{M}$, as described by Thess *et al.* (1997) for Marangoni convection. In real space the velocity is expressed as

$$\mathbf{v}_H(\mathbf{r}) = -\frac{1}{4\pi} \iint \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} M(\mathbf{r}') d^2\mathbf{r}' \equiv \mathcal{V}_0[M(\mathbf{r})], \tag{4.2}$$

defining the non-local operator \mathcal{V}_0 relating the velocity field to the M field. This integral (4.2) as well as various other integrals that will be used in the next paragraphs must be understood in terms of Cauchy principal value determination. The kernel $\mathbf{K}(\mathbf{r}) = -[1/(4\pi)]\mathbf{r}/|\mathbf{r}|^3 = [1/(4\pi)]\nabla(1/r)$ is formally identical to the Green function for the Laplacian with forcing at the boundary. The operator \mathcal{V}_0 is linear and is invariant for a change of scale $\mathbf{r} \rightarrow \beta\mathbf{r}$, reflecting the fact that convection in a half-space (infinite layer) has no internal length scale,

$$\mathcal{V}_0[\alpha M] = \alpha\mathcal{V}_0[M], \quad \mathcal{V}_0[M(\beta\mathbf{r})] = \mathcal{V}_0[M(\mathbf{r})]. \tag{4.3}$$

In the opposite limiting case $\sigma \gg 1$, the closure relation in Fourier space is

$\hat{v}_H(\mathbf{k}) = (i\mathbf{k}/4\sigma k^2)\hat{M}(\mathbf{k})$. By multiplying this equation by $i\mathbf{k}$, we deduce that

$$\nabla_H \cdot \mathbf{v}_H = -\frac{M}{4\sigma}, \quad (4.4)$$

which has the solution

$$\mathbf{v}_H(\mathbf{r}) = -\frac{1}{8\pi\sigma} \iint \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} M(\mathbf{r}') d^2\mathbf{r}' \equiv \mathcal{V}_\infty[M(\mathbf{r})]. \quad (4.5)$$

Like equation (4.2), the velocity field is related to M by a non-local operator, \mathcal{V}_∞ . This operator is also linear in M , but it is no longer scale invariant because the convection system has now an internal length scale as σ has the dimension of a length,

$$\mathcal{V}_\infty[\alpha M] = \alpha \mathcal{V}_\infty[M], \quad \mathcal{V}_\infty[M(\beta\mathbf{r})] = \beta \mathcal{V}_\infty[M(\mathbf{r})]. \quad (4.6)$$

The reverse transformation that gives the moment as a function of the surface velocity is easy to derive in the real space whatever the value of σ . The closure law (3.3) can also be written

$$\hat{M}(\mathbf{k}) = -2i \frac{1 + 2\sigma k}{k} \mathbf{k} \cdot \hat{\mathbf{v}}_H(\mathbf{k}), \quad (4.7)$$

which corresponds to

$$M(\mathbf{r}) = -4\mathcal{V}_0[\mathbf{v}(\mathbf{r})] - 4\sigma \nabla \cdot \mathbf{v}_H \quad (4.8)$$

(note that the operator \mathcal{V}_0 according to its definition (4.2) either maps a scalar to a vector or a vector to a scalar, just like the operator ∇). Equation (4.8) means that the vertical moment of the temperature across the thermal boundary layer can be estimated from the surface velocity on top of the convective medium.

4.2. Examples of moment–velocity closure relationships

In one dimension, i.e. assuming that \mathbf{v}_H and M are only functions of x , the expression (4.2) can be integrated in y , which leads to

$$v_H(x) = \frac{1}{2} \mathcal{H}[M(x)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{M(x')}{x' - x} dx'. \quad (4.9)$$

where the symbol \mathcal{H} stands for the Hilbert transform (Erdélyi 1954). This operator has the same scaling properties (4.3) as its two-dimensional counterpart.

Hilbert transforms of particular functions are tabulated in mathematical handbooks (Erdélyi 1954), see also table 1 of Thess *et al.* (1997). As a simple example of physical interest, the velocity induced by the field

$$M(x) = \frac{1}{1 + x^2} \quad (4.10)$$

is

$$v_H(x) = -\frac{x}{2(1 + x^2)}. \quad (4.11)$$

If the vertical temperature moment is restricted to a singular line, i.e. $M(x) = \delta(x)$, the induced velocity is simply

$$v_H(x) = -\frac{1}{2\pi x}. \quad (4.12)$$

These two solutions are depicted in figure 4(b).

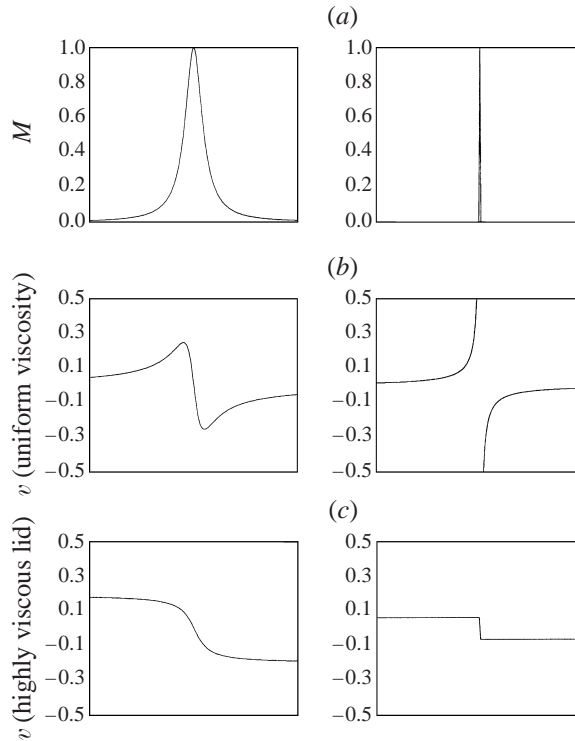


FIGURE 4. Examples of one-dimensional closure relationships, for $M(x) = 1/(1 + x^2)$ (left-hand column) and $M(x) = \delta(x)$ (right-hand column): (a) M , (b) the corresponding velocity with a uniform viscosity ($\sigma = 0$), and (c) the velocity with a highly viscous lid ($\sigma \gg 1$).

In the case $\sigma \gg 1$, (4.5) integrated in y leads to

$$v_H(x) = \frac{1}{8\sigma} \left(\int_{-\infty}^x M(x') dx' + \int_{\infty}^x M(x') dx' \right). \tag{4.13}$$

The velocity induced by the M field (4.10) is

$$v_H(x) = \frac{1}{8\sigma} \arctan(x), \tag{4.14}$$

represented in figure 4(c). As expected, it is smoother than in the case $\sigma = 0$. The velocity induced by a singular line $M(x) = \delta(x)$ is the step function (written with the Heaviside function H),

$$v_H(x) = \frac{1}{4\sigma} \left(\frac{1}{2} - H(x) \right). \tag{4.15}$$

We see that when the role of the highly viscous lid becomes important, a uniform velocity is induced on each side of the singular temperature source.

We can also express the closure law in a two-dimensional axisymmetric geometry appropriate to describe plumes. If the vertical temperature moment is restricted to a singular point, $M(r) = \delta(r)$, the equations (4.2) and (4.5) become

$$v_r(r) = -\frac{1}{4\pi r^2}, \tag{4.16}$$

when $\sigma = 0$, and with a highly viscous lid at the surface, $\sigma \gg 1$,

$$v_r(r) = -\frac{1}{8\pi\sigma} \frac{1}{r}. \quad (4.17)$$

As in the Cartesian one-dimensional case, the presence of a highly viscous lid increases the effect of a perturbation at large distances.

5. Finite-time singularities

Having understood the first stages of the development of an instability we must now study the behaviour of the finite-time singularities. When a singularity occurs, M goes to $+\infty$ and the last term of equation (3.2), corresponding to the secular diffusive increase of the thermal boundary layer, can be safely neglected. In this case we can search for solutions with separate variables of the form

$$M(\mathbf{r}, t) = (t_s - t)^a F(\boldsymbol{\chi}), \quad (5.1)$$

where

$$\boldsymbol{\chi} = \frac{(\mathbf{r} - \mathbf{r}_s)}{(t_s - t)^b}. \quad (5.2)$$

The constants a and b are the critical exponents of the singularity that occurs at position \mathbf{r}_s and time t_s .

As the scaling properties of the operator that relates the moment to the horizontal velocity depends on the surface viscosity σ , we first present in detail the case $\sigma = 0$ and then discuss the role of the surface viscosity. When $\sigma = 0$, the operator relating moment and velocity is \mathcal{V}_0 (4.3) and by plugging (5.1) into (3.2) we get

$$\begin{aligned} -aF + b\boldsymbol{\chi} \cdot \nabla_H F - (t_s - t)^{-2b+1} \nabla_H^2 F \\ - (t_s - t)^{a-b+1} (\mathcal{V}_0[F] \cdot \nabla_H F + 2F \nabla_H \cdot \mathcal{V}_0[F]) = 0. \end{aligned} \quad (5.3)$$

Choosing $a = -\frac{1}{2}$ and $b = \frac{1}{2}$, this equation becomes time independent, and

$$M(\mathbf{r}, t) = \frac{1}{(t_s - t)^{1/2}} F \left(\frac{\mathbf{r} - \mathbf{r}_s}{(t_s - t)^{1/2}} \right), \quad (5.4)$$

is a solution of equation (3.2) when F verifies

$$F + \boldsymbol{\chi} \cdot \nabla_H F - 2\mathcal{V}_0[F] \cdot \nabla_H F - 4F \nabla_H \cdot \mathcal{V}_0[F] - 2\nabla_H^2 F = 0. \quad (5.5)$$

The self-similar solution $M(\mathbf{r}, t)$ has a maximum divergence as $(t_s - t)^{-1/2}$, and a width decreasing as $(t_s - t)^{1/2}$. For a line singularity (one-dimensional geometry), with boundary conditions $\nabla_H F = 0$ on the instability and F vanishing at large distance, we find numerically that the solution for F is unique (its shape but also its amplitude). In the two-dimensional axisymmetric case, another solution is obtained as the differential operators entering equation (5.5) are different, but again this axisymmetric solution with its shape and its amplitude is univocally obtained by (5.5).

Instead of trying to solve the difficult differential equation (5.5), we have computed numerically in the one-dimensional case $M(x, t)$ for various initial conditions, measured the position x_s and time t_s for the first singularity and plotted the quantity $(t - t_s)^{1/2} M(x, t)$ as a function of $(x - x_s)/(t_s - t)^{1/2}$ for times t close to t_s . The results depicted in figure 5(a) for four different initial conditions show a universal shape, when properly scaled. Of course, when plotted with a logarithmic scale (b), the presence of other weaker growing singularities far from the main one is clearly shown

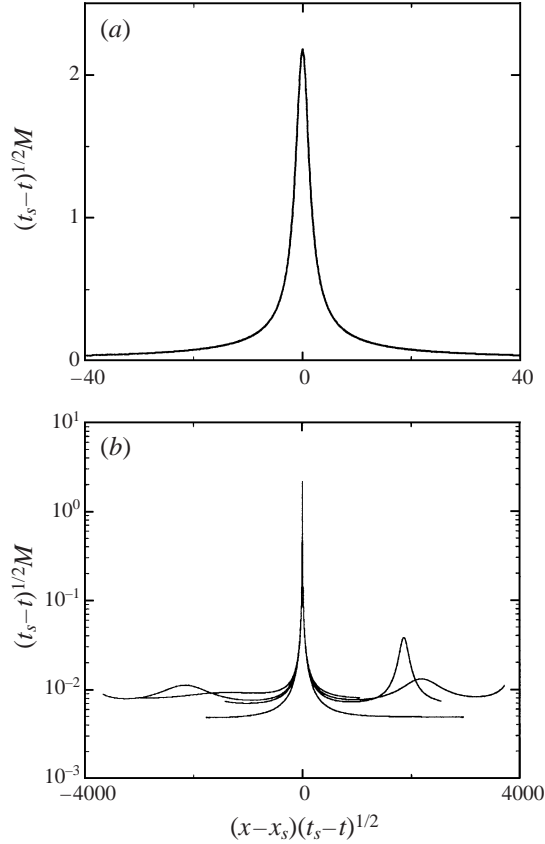


FIGURE 5. Universal shape of the one-dimensional singularity in the case of a constant viscosity. (a) Four different numerical experiments have been performed and have been rescaled. (b) A vertical logarithmic scale and a wider spatial extent shows the presence of other singularities in formation.

as secondary peaks. We also run two-dimensional axisymmetric cases, and similar results are obtained, i.e. the same critical exponents and a unique F solution.

We can use a similar procedure in the case of a highly viscous lid $\sigma \gg 1$, but with the scaling relationship (4.6). This also leads to self-similar solutions of the form

$$M(x, t) = \frac{\sigma}{t_s - t} G\left(\frac{x - x_s}{(t_s - t)^{1/2}}\right), \quad (5.6)$$

where G is another universal function when the geometry, one-dimensional or two-dimensional, is chosen. The maximum now increases as $(t_s - t)^{-1}$, faster than in the case without a lid, and its width decreases as $(t_s - t)^{1/2}$.

Figure 6 depicts the same set as in figure 1 but now the behaviour of $\log(\text{Max}_x M(x, t))$ is plotted as a function of $\log(t - t_s)$. We also show the case with a highly viscous lid ($\sigma \gg 1$). The theoretical laws with exponents $-\frac{1}{2}$ and -1 are closely followed.

The behaviour of M far from a growing instability can be described analytically. In equation (5.5), the terms associated with cross-products between velocity and moment ($-2\mathcal{V}_0[F] \cdot \nabla F - 4F \nabla \cdot \mathcal{V}_0[F]$) are smaller than the first two terms if $\mathcal{V}_0[F] \ll \chi$. In this case the solutions of (5.5) are $F(x) \sim A/x$ in the one-dimensional case and $F(r) \sim B/r$ in the axisymmetric case. The proportionality constants A and B are not

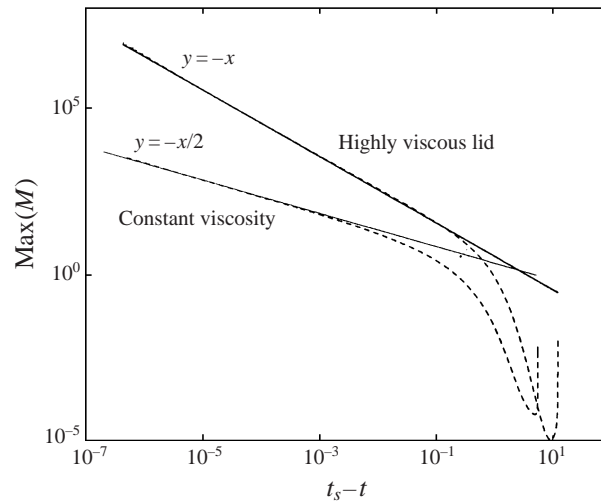


FIGURE 6. Time behaviour of the instability near the singularity in the two cases of constant viscosity and of a highly viscous lid. Because the abscissa is $t_s - t$, the singularity now evolves from right to left.

	Uniform viscosity	Highly viscous lid
Line singularity	$-\frac{A \ln(x(t_s - t)^{-1/2})}{2\pi x}$	$-\frac{A \ln(x(t_s - t)^{-1/2})}{4\sigma(t_s - t)}$
Axisymmetric singularity	$-\frac{B}{2r}$	$-\frac{B}{4\sigma(t_s - t)}$

TABLE 1. Asymptotic velocity at large distance of a singularity.

arbitrary but are univocally determined by the nonlinearity of the differential equation close to the singularity. From the shape of the M singularity at large distance, we can use the closure laws (4.2) and (4.5) to deduce the velocity far from the singularity. After some algebra we obtain the asymptotic behaviours summarized in table 1.

The results of table 1 indicate that singularities interact at very large distances and potentially at an infinite distance (highly viscous lid), due to the transport by their induced velocity. Although in figure 4 the velocity induced by a temperature moment decreasing asymptotically as $1/x^2$ remains finite, the natural one-dimensional singularity in a fluid with a highly viscous lid has a temperature moment decreasing only as $1/x$ and therefore induces a velocity diverging as $\ln(x)$. Numerically, the forced periodicity imposed by the use of fast Fourier transforms makes the verification of these laws difficult.

6. Developed convection

6.1. Regularization of the singularity

When plumes develop, the temperature heterogeneity θ leaves the boundary layer, so we cannot assume anymore that θ decreases to zero in the range of integration used to get (2.43). Taking into account the value of $\theta(Z)$ at the upper bound of integration

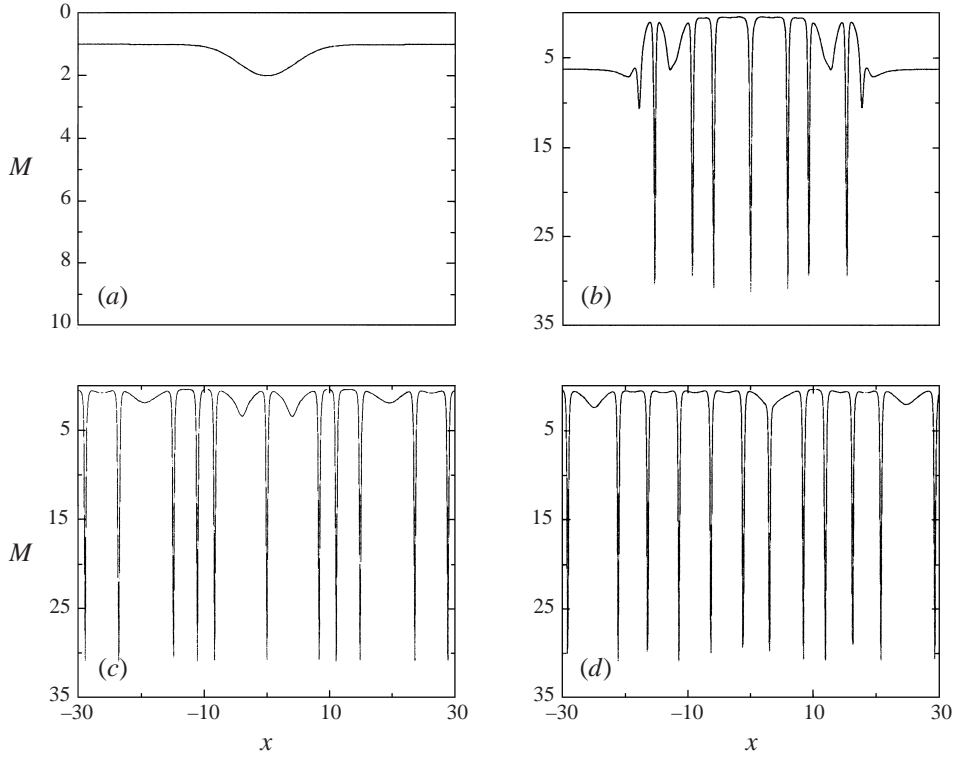


FIGURE 7. Growth and dynamics of one-dimensional instabilities, $M(x, t)$ as a function of x at four different times t : (a) $t = 0$, (b) 5, (c) 10, (d) 50.

Z , we generalize (2.43) to

$$\frac{\partial M}{\partial t} + (\mathbf{v}_H \cdot \nabla_H)M + (2M + Z^2\theta(Z))\nabla_H \cdot \mathbf{v}_H = \kappa \nabla_H^2 M - \kappa(\theta_S - \theta(Z)). \quad (6.1)$$

We need to close this equation to determine $\theta(Z)$ as a function of the dynamical variables. At small times, $\theta(Z) = 0$, and we recover (2.43). In the opposite case of a plume extending beyond the depth Z , the temperature becomes nearly uniform over the depth Z , so we have $Z^2\theta(Z) \simeq Z^2\theta_S \simeq -2M$; thus the source terms disappear in (6.1). We propose an heuristic fit between these two extremes, by writing $(2M + Z^2\theta(Z))\nabla_H \cdot \mathbf{v}_H = M[1 - \tanh(M - M_{max})]\nabla_H \cdot \mathbf{v}_H$, where $M_{max} = -Z^2\theta_S/2$. This provides a regularizing mechanism for the plumes. A corresponding heat injection should be introduced in the interior, providing an additional source of motion. We expect this motion to be at fairly large scales, with a weak influence on the plume dynamics, and we have neglected this modification of the interior in the present study. We also neglect $\theta(Z)$ on the right-hand side of equation (6.1) as this diffusion term is negligible in plumes in comparison with the other effects.

6.2. Developed convection in the one-dimensional case

With this regularization mechanism, our model can go beyond the first finite-time singularity and a much more complex dynamics is obtained. Figure 7 shows the evolution of $M(x)$ at four different times (one-dimensional case, with uniform viscosity, $\sigma = 0$). In this simulation we have chosen to cut the singularities around $M_{max} = 30$. The progressive destabilization of the boundary layer starts from panel (a) ($t = 0$)

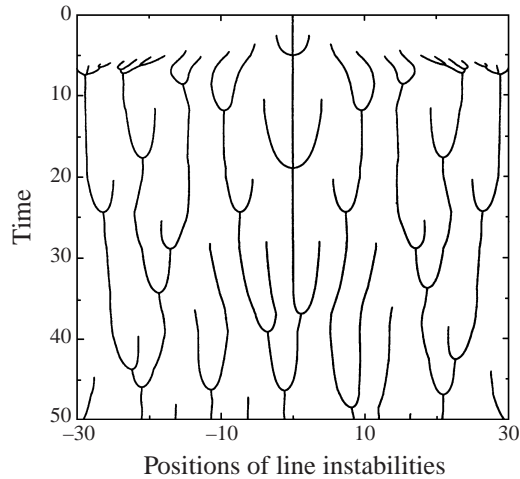


FIGURE 8. Positions of the line instabilities of figure 7 as functions of time.

where the arbitrary initial moment has a Gaussian shape. The reader must however realize that the equivalent boundary layer thickness, $(M)^{1/2}$, is of the same order as the horizontal scale.

The first singularities start near the maximum of M (b) in agreement with the stability analysis. The boundary layer is then destabilized everywhere. For small times (b , c) the symmetry of the initial conditions is preserved. At larger times, (d), the symmetry is broken by the birth of new instabilities.

In order to more clearly understand the initiation and interactions of instabilities, figure 8 depicts the position of the peaks as a function of time for the same simulation as in figure 7. At the beginning, a large number of peaks is produced; this number then reduces to about 10 peaks. This reduction follows the reduction of the average boundary layer thickness. This corresponds qualitatively to the fact that the wavelength of the most unstable perturbation in the marginal stability study increases when the average thickness decreases.

The various peaks attract each other. When two peaks merge, a new peak appears in the space left empty. Because of this chaotic behaviour, the pattern loses its symmetry after a time larger than 30 in this simulation. The number of peaks is then rather constant on average and about one peak every 2π (i.e. about 10 peaks in this simulation where the abscissae goes from -10π to 10π).

The characteristic parabolic shape of the trajectories of two interacting peaks is easy to understand at least qualitatively following a method originally applied to Marangoni convection (Thess *et al.* 1995, 1997). Each peak induces in its vicinity an attracting velocity $v_H(x)$, transporting its surroundings, so that the distance X between them varies as

$$\frac{dX}{dt} = 2v_H(X). \quad (6.2)$$

If we assume that the two peaks are close enough to interact but far enough apart so that each one induces a velocity as if it were alone (vortex dynamics, Aref 1983), then the attracting velocity can be computed. For one dimension and $\sigma = 0$, the closure law needs the introduction of the Hilbert transform of the moment of the instability (4.9). This closure law can be written in a different form, that has the advantage of

using defined integrals rather than Cauchy improper integrals,

$$v_H(X) = \frac{1}{2\pi} \int_0^\infty \frac{M(X+x') - M(X-x')}{x'} dx'. \quad (6.3)$$

The numerator varies much more rapidly around $x' = X$ than the denominator and far from this point the numerator is very small. Therefore, we can approximate the $1/x'$ under the integral sign by $1/X$ and, using the parity of $M(x)$, we get

$$v(X) \sim \frac{1}{2\pi X} \int_0^\infty M(X+x') - M(X-x') dx' = -\frac{1}{2\pi X} \int_{-X}^X M(x) dx. \quad (6.4)$$

When the heads of the instabilities are cut beyond a limited range, this last integral is roughly a constant I_0 and thus

$$\frac{dX}{dT} = -\frac{I_0}{\pi X}, \quad (6.5)$$

or

$$X^2 = X_0^2 - 2\frac{I_0}{\pi}t, \quad (6.6)$$

where X_0 is the initial distance between peaks. This last equations explains the parabolic trajectories occurring when two peaks collapse. When the instabilities are not limited by a maximum size M_{max} , the distance between two isolated peaks can still be closely fitted for a limited range of distances by a parabolic law although we know that the integral in equation (6.4) slowly diverges as $\ln(X/(t_s - t)^{1/2})$ in agreement with table 1.

We carefully verified equations (6.4)–(6.6) in the presence of two isolated peaks. However, this situation is not stable and soon other peaks appear. Quantitatively, we can extract from figure 8 a value of I_0 that gives the best fit to the behaviour of the trajectories when they merge. We found a value of about 4 times smaller than the integral of each singularity. This discrepancy is not understood although we think that it is not produced by the approximation of (6.4) but rather by a collective effect due to the other instabilities interacting at large range. It is remarkable that the same parameter I_0 seems to characterize all the interactions of two peaks.

6.3. Developed convection in the two-dimensional case

In two dimensions, we also run our program starting from initial conditions

$$\begin{aligned} M(x, y, 0) = & 4 + \cos(4\pi y/L) \cos(2\pi x/L) \\ & + \sin(2\pi x/L) \sin(2\pi y/L) + \cos(4\pi y/L) \sin(6\pi x/L) \end{aligned} \quad (6.7)$$

where the size of the box L is 5. This functional dependence (except for the mean value equal to 4) was used by Thess *et al.* (1997) in their study of Marangoni convection. We verified that our program exactly reproduces their results when the term that contains $\nabla_H \cdot \mathbf{v}_H$ is suppressed in equation (6.1), and when the thermal diffusivity is small. In the case of Rayleigh–Bénard convection with only cooling from above, figure 9 depicts various results as a function of time. We only show the case $\sigma = 0$ in this simulation. As seen for the closure relationship (2.41), increasing σ tends to smooth the velocity field and eases the computation. As in one-dimensional geometry, we first observe the increase of the boundary layer thickness and the growth of instabilities that keep the geometry of the initial conditions (figure 9a, b). The cold plumes and sheets start attracting each other according to the mechanism already

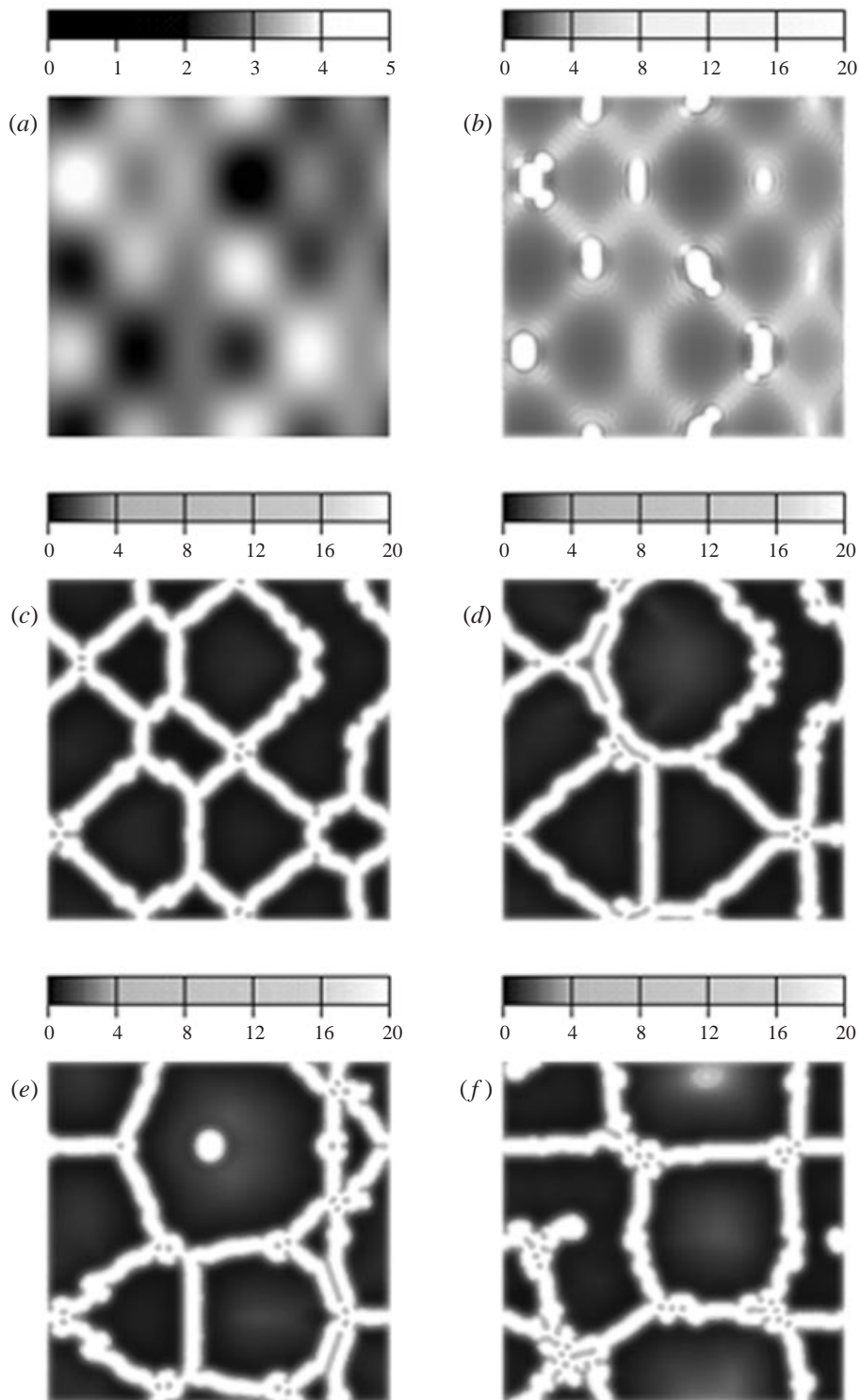


FIGURE 9. Rayleigh–Bénard convection with uniform viscosity for a fluid cooled from above: (a) the initial condition, (b) $t = 1$, (c) $t = 4$, (d) $t = 8$, (e) $t = 13.5$ and (f) $t = 30$.

discussed (*c*, *d*). The initial conditions are still apparent in the pattern of convection. The merging of some instabilities liberates enough space for a new instability to occur as a plume structure in the middle of a roughly hexagonal cell (*e*). At a later stage, the memory of the initial geometry is totally lost (*f*) but the topological characteristics of the convection pattern (i.e. the number of cells, the length of cold downwellings, etc.) remain the same.

6.4. Nusselt–Rayleigh relationship

Usually in a convection experiment, i.e. a liquid tank of height L , where a temperature difference ΔT is imposed between the surface and the bottom, the convective activity can be estimated by two non-dimensional numbers. The first is the Rayleigh number Ra , the normalized temperature difference, and the second is the Nusselt number Nu , the heat flux Q normalized by the heat flux that would occur by pure conduction. These two numbers are

$$Ra = \frac{\alpha \rho_0 g \Delta T L^3}{\eta_0 \kappa}, \quad (6.8)$$

and

$$Nu = \frac{LQ}{\kappa \rho_0 C_p \Delta T} \quad (6.9)$$

(the thermal conductivity is $\kappa \rho_0 C_p$ where C_p is the heat capacity). These two numbers are related by the dynamics, and we expect a model of convection to yield this relationship.

In our model, the flow is entirely driven by boundary layer effects, with no influence of the thickness L , which is supposed very large. Then the only possible relationship, which does not depend on L , is

$$Nu = a Ra^{1/3}, \quad \text{with } a = \frac{Q}{\kappa \rho_0 C_p \Delta T} \left(\frac{\eta_0 \kappa}{\alpha \rho_0 g \Delta T} \right)^{1/3} \quad (6.10)$$

In our model, Nu and $Ra^{1/3}$ are both infinite, being both proportional to the thickness L , but the constant a is well defined and can be calculated as follows.

The average surface heat flow is given by

$$Q = -\kappa \rho_0 C_p \frac{1}{S} \int_S \frac{\partial \theta}{\partial z} dS, \quad (6.11)$$

where the integration is performed at the surface S of the convective fluid. Our model deals with the variable M , the first moment of the temperature in the boundary layer, and we need to make some assumptions about the vertical temperature profile to calculate the heat flux (6.11). At short times, temperature satisfies the diffusion equation, and varies with z like a complementary error function; therefore $\theta = \theta_S \operatorname{erfc}(z/H)$ where H is the thickness of the boundary layer and $\partial_z \theta = -2\theta_S/(\pi^{1/2}H)$. Using (2.44) that relates M and H ($M = -\theta_S H^2/4$), we then get

$$Q = \kappa \rho_0 C_p \frac{1}{S} \int_S \frac{(-\theta_S)^{3/2}}{(\pi M)^{1/2}} dS. \quad (6.12)$$

This last dimensional expression can be written with a non-dimensional M as

$$Q = -\kappa \rho_0 C_p \theta_S \left(\frac{-\alpha \rho_0 g \theta_S}{\eta_0 \kappa} \right)^{1/3} \frac{1}{S} \int_S \frac{1}{(\pi M)^{1/2}} dS, \quad (6.13)$$

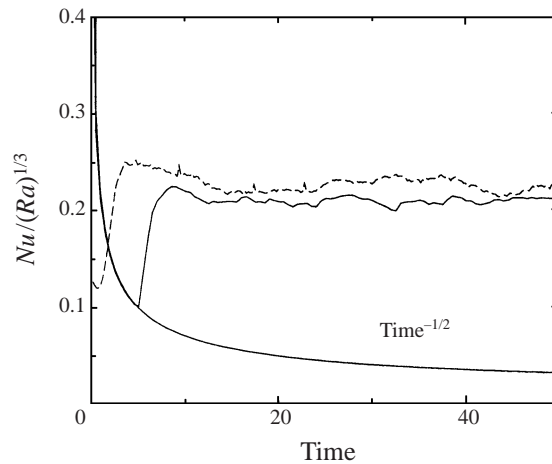


FIGURE 10. Evolution of $Nu/Ra^{1/3}$ as a function of time, in one-dimensional (solid line) and two-dimensional (dashed line) simulations.

or by the introduction of a dummy length L , and noting that $\Delta T = -2\theta_s$ since we need to symmetrize the system with two boundary layers to fit with the usual Rayleigh–Bénard configuration,

$$Nu = \left(\frac{1}{S} \int_S \frac{1}{2^{4/3}(\pi M)^{1/2}} dS \right) Ra^{1/3}, \quad (6.14)$$

in agreement with the functional form of equation (6.10).

Figure 10 depicts the average over the surface of $(2)^{-4/3}(\pi M)^{-1/2}$, as a function of time, in the one-dimensional case (solid line), and two-dimensional case (dashed line), these two simulations being performed with $M_{max} = 30$. In the one-dimensional case, the initial conditions are simply a very small M ; in the two-dimensional case we use the same boundary conditions as in figure 9. In the one-dimensional case, at the beginning, the thermal boundary thickness is very small and grows by simple diffusion following a $t^{-1/2}$ law. This behaviour is not as clearly seen for the two-dimensional case, as we start in a regime where the boundary layer is already unstable. When the convection starts the heat flow increases and then stabilizes around $Nu/Ra^{1/3}$ close to 0.22.

The mean value of $Nu/Ra^{1/3}$ is slightly dependent on the choice of M_{max} (figure 11): it increases from 0.165 for $M_{max} = 10$ to 0.235 for $M_{max} = 40$ (one-dimensional case). The numerical experiment may indicate an asymptotic value for very large M_{max} . However there may be no truly asymptotic value, which would reflect some departure from the $Ra^{1/3}$ law. Indeed we may expect that the thickness $\sim M^{1/2}$ associated with the maximum possible value of M scales with the thickness L .

Our Rayleigh–Nusselt relationship can be compared with results found in the literature. For convection at infinite Prandtl number, and heated from below, simple boundary layer models (Turcotte & Oxburg 1967) give $Nu \sim 0.294 Ra^{1/3}$. Two- and three-dimensional numerical simulations (McKenzie, Roberts & Weiss 1974; Travis, Olson & Schubert 1990; Tackley 1996; Sotin & Labrosse 1999) and laboratory experiments at very high Prandtl number (Giannandrea & Christensen 1993; Manga & Weeraratne 1999) provide similar values (although the exponent seems smaller than 1/3). Laboratory experiments by Goldstein, Chiang & See (1990) give $Nu \sim 0.066 Ra^{1/3}$ but for no-slip boundary conditions and a moderate Prandtl number.

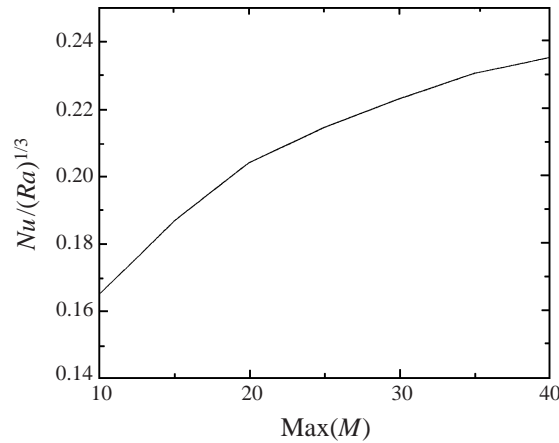


FIGURE 11. Dependence of the Nusselt number on the maximum moment M_{max} .

7. Conclusion

We have shown that the equations of three-dimensional convection in the limit of high Rayleigh and Prandtl numbers can be reduced to two-dimensional equations written at the surface of the convective fluid. These equations express how the moment of the temperature through the thermal boundary layer is transported and modified by the surface two-dimensional velocity field. They are universal, i.e. they do not contain any physical parameters. This reduction from three to two dimensions provides an elegant tool to study the initiation of thermal plumes and line instabilities.

Various generalizations of the present theory can be done in a straightforward manner. The role of a depth-dependent viscosity in the thermal boundary layer is controlled by a parameter σ . However we could have also considered depth-dependent viscosity variations in the deep interior. As an example the viscosity of silicated planets increases significantly with depth. This can be very easily taken into account by a modification of the closure law. Another modification of the closure law could allow us to study the interaction of a top boundary layer with a bottom boundary layer. In this case two transport equations would have to be coupled through two closure laws relating linearly in the spectral space the surface velocity of each boundary layer to the two temperature moments.

Apart from its utility in understanding the development of instabilities, their interactions, the similarities and differences between Marangoni convection and thermal convection, our approach suggests a method to study a very important geophysical problem, namely the interactions between plate tectonics and the underlying high-Rayleigh-number convection. In our planet, the rheology of the surface boundary layer is highly nonlinear and therefore the present theory does not apply. However, we have purposely used as long as possible stresses rather than velocities in deriving the equations of this paper in order to distinguish what is related to the assumption of a Newtonian rheology and what is perfectly general. The relationships between stresses and temperature moment (2.20), (2.21) and the transport equation (2.43) are independent of the rheology. This means we are able to reduce the three-dimensional hydrodynamic problem of plate tectonics to two-dimensional surface equations even in the case of a very complex relationship between stresses and velocities. Of course, in

this case the equivalent of the closure relationship will be only obtained numerically and a vertical vorticity would potentially be excited.

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